Lesson 8. Solving Dynamic Programs with networkx

Overview

- In this lesson, we'll revisit a few examples of dynamic programs and solve them with `networkx`.

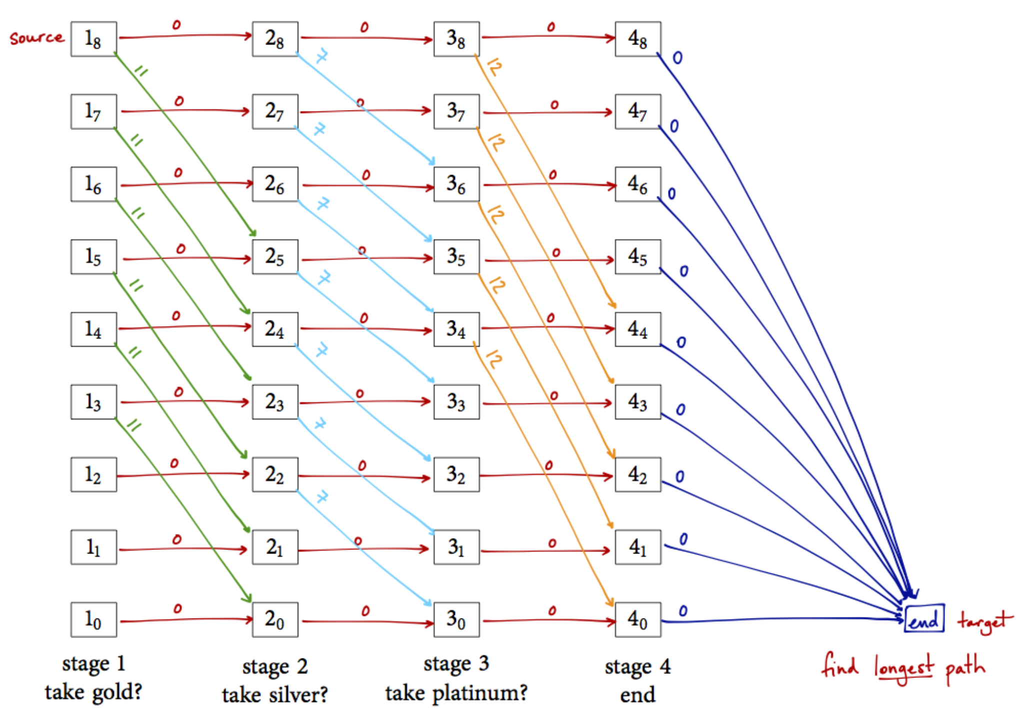
The knapsack problem, revisited

You are a thief deciding which precious metals to steal from a vault:

	Metal	Weight (kg)	Value
1	Gold	3	11
2	Silver	2	7
3	Platinum	4	12

You have a knapsack that can hold at most 8 kg. If you decide to take a particular metal, you must take all of it. Which items should you take to maximize the value of your theft?

- Recall that we formulated this problem as a dynamic program with the following longest path representation:
    - Stage t represents the decision to take item t ($t = 1, 2, 3$), or the end of the decision-making process ($t = 4$).
    - Node t_n represents having n kgs left in knapsack at stage t ($n = 0, 1, \ldots, 8$).



DP for knapsack example

- We know how to solve shortest/longest path problems using `networkx`, so we can apply the same ideas here.
- There is a Python data structure that makes this a little easier...

Tuples

- A **tuple** is like a list, except once it's been defined, it cannot be changed.
- A tuple is written as a sequence of comma-separated items between *round* brackets. For example:

```
In [2]: # Define a tuple corresponding to taking
        # silver with 5 kgs left in the knapsack
        stage = (2, 5)
```

- Tuples are ideal for things like names of nodes — things that you want to make permanent and not accidentally change.

Back to the knapsack problem...

- We can use a tuple to represent the name of each node in our dynamic program, since each node's name has two distinct parts: the stage and the state.
- Before we do anything, we need to import `networkx` and `bellmanford`:

```
In [3]: import networkx as nx
        import bellmanford as bf
```

- Let's begin by creating an empty graph:

```
In [4]: # Create empty graph
        G = nx.DiGraph()
```

- Next, let's add the stage-state nodes to the graph, using `for` loops. Remember that `range(a, b)` iterates over the integers `a, a + 1, ..., b - 1`.

```
In [5]: # Add the stage-state nodes
        for t in range(1, 5):
            for n in range(0, 9):
                G.add_node((t, n))
```

- We also need to add the special "end" node:

```
In [6]: # Add the end node
        G.add_node("end")
```

- Now we need to add the edges.
- There are a lot of them, so we'll want to use some for loops.
- The best way to use for loops depends on the shortest/longest path representation of the DP.
- For example, looking above, we can add all the red edges of length 0 — corresponding to not taking the item — in one fell swoop, like this:

```
In [7]: # Add edges corresponding to not taking an item
        # (red edges of length 0)
        for t in range(1, 4):
            for n in range(0, 9):
                G.add_edge((t, n), (t + 1, n), length=0)
```

- Next, we can add the green edges of length 11, corresponding to taking item 1 (gold). Don't forget our DP is a *longest* path problem!

```
In [8]: # Add edges corresponding to taking item 1
        # (green edges of length 11)
        for n in range(3, 9):
            G.add_edge((1, n), (2, n - 3), length=-11)
```

- We can do something similar for the light blue and orange edges as well:

```
In [9]: # Add edges corresponding to taking item 2
        # (light blue edges of length 7)
        for n in range(2, 9):
            G.add_edge((2, n), (3, n - 2), length=-7)

        # Add edges corresponding to taking item 3
        # (orange edges of length 12)
        for n in range(4, 9):
            G.add_edge((3, n), (4, n - 4), length=-12)
```

- Finally, we can add the edges from the last stage nodes to the special "end" node:

```
In [10]: # Add edges from stage 4 to the end node
         for n in range(0, 9):
             G.add_edge((4, n), "end", length=0)
```

- Now, we can solve the dynamic program using the Bellman-Ford algorithm, just as before:

```
In [11]: # Solve DP by solving its shortest path representation using Bellman-Ford
         length, nodes, negative_cycle = bf.bellman_ford(G, source=(1, 8), target="end",
         weight="length")
         print("Shortest path length: {0}".format(length))
         print("Shortest path: {0}".format(nodes))
```

```
Shortest path length: -23
Shortest path: [(1, 8), (2, 5), (3, 5), (4, 1), 'end']
```

Interpreting the output

- What is the maximum value we can carry in the knapsack?

The maximum value we can carry in the knapsack is 23, the negative of the shortest path length.

- Which items should we take to obtain this maximum value?

According to the edges in the shortest path, we should take the gold and platinum, but not the silver.

Practice makes perfect — on your own

- Here are a two more examples of dynamic programs we modeled in a previous lesson. Solve them using `networkx` and interpret the output.

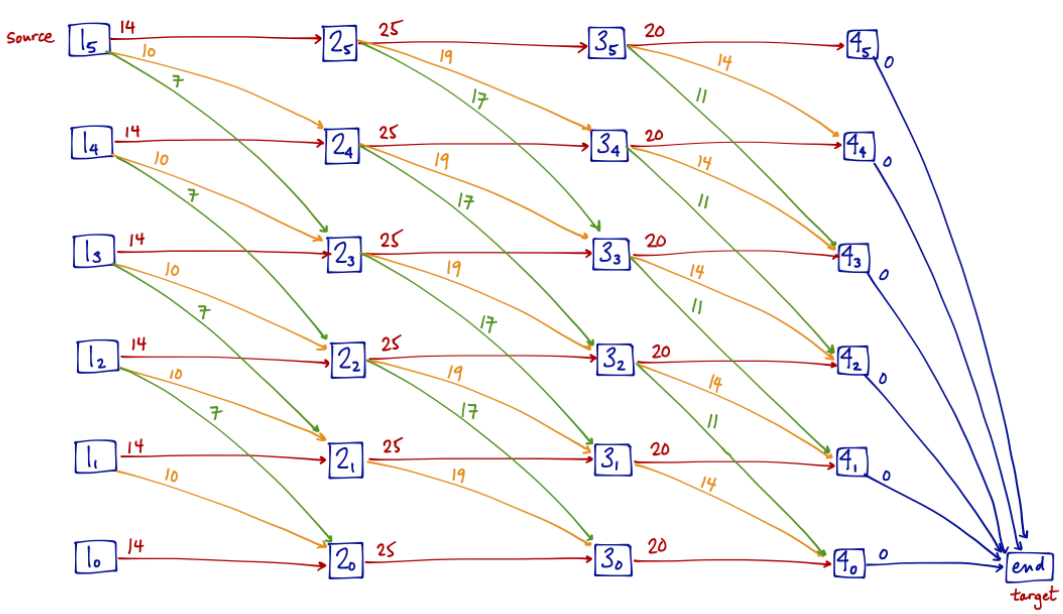
Assigning patrol cars to precincts

The Simplexville Police Department wants to determine how to assign patrol cars to each precinct in Simplexville. Each precinct can be assigned 0, 1, or 2 patrol cars. The number of crimes in each precinct depends on the number of patrol cars assigned to each precinct:

Precinct	0 patrol cars	1 patrol cars	2 patrol cars
1	14	10	7
2	25	19	17
3	20	14	11

The department has 5 patrol cars. The department's goal is to minimize the total number of crimes across all 3 precincts.

- We formulated this problem as a dynamic program with the following shortest path representation:
    - Stage t represents the decision to assign patrol cars to precinct t $(t = 1, 2, 3)$ or the end of the decision-making process $(t = 4)$.
    - Node t_n represents having n patrol cars left at stage t $(n = 0, 1, \ldots, 5)$.



DP for patrol car example

```
In [12]: # Solve this DP using networkx here
         # Create empty graph
         G = nx.DiGraph()

         # Add the stage-state nodes
         for t in range(1, 5):
```

```python
            for n in range(0, 6):
                G.add_node((t, n))

        # Add the end node
        G.add_node("end")

        # Add edges corresponding to adding 0 patrol cars - red edges
        for n in range(0, 6):
            # precinct 1: length 14
            G.add_edge((1, n), (2, n), length=14)

            # precinct 2: length 25
            G.add_edge((2, n), (3, n), length=25)

            # precinct 3: length 20
            G.add_edge((3, n), (4, n), length=20)

        # Add edges corresponding to adding 1 patrol car - orange edges
        for n in range(1, 6):
            # precinct 1: length 10
            G.add_edge((1, n), (2, n - 1), length=10)

            # precinct 2: length 19
            G.add_edge((2, n), (3, n - 1), length=19)

            # precinct 3: length 14
            G.add_edge((3, n), (4, n - 1), length=14)

        # Add edges corresponding to adding 2 patrol cars - green edges
        for n in range(2, 6):
            # precinct 1: length 7
            G.add_edge((1, n), (2, n - 2), length=7)

            # precinct 2: length 17
            G.add_edge((2, n), (3, n - 2), length=17)

            # precinct 3: length 11
            G.add_edge((3, n), (4, n - 2), length=11)

        # Add edges from last stage to the end node
        for n in range(0, 6):
            G.add_edge((4, n), "end", length=0)

        # Solve DP by solving its shortest path representation using Bellman-Ford
        length, nodes, negative_cycle = bf.bellman_ford(G, source=(1, 5), target="end",
        weight="length")
        print("Shortest path length: {0}".format(length))
        print("Shortest path: {0}".format(nodes))
```

```
Shortest path length: 37
Shortest path: [(1, 5), (2, 3), (3, 2), (4, 0), 'end']
```

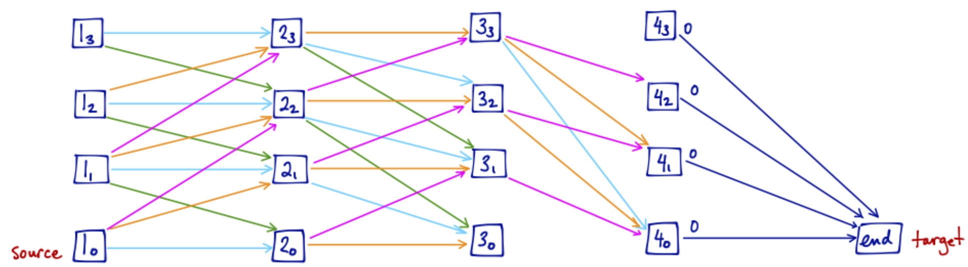
- The minimum number of crimes as a result of assigning the 5 patrol cars to the 3 precincts is 37, the shortest path length.
- To achieve this minimum number of crimes, assign 2 patrol cars to precinct 1, 1 patrol car to precinct 2, and 2 patrol cars to precinct 3.

Inventory management

The Dijkstra Brewing Company is planning production of its new limited run beer, Primal Pilsner. The company must supply 1 batch next month, then 2 and 4 in successive months. Each month in which the company produces the beer requires a factory setup cost of \$5,000. Each batch of beer costs \$2,000 to produce. Batches can be held in inventory at a cost of \$1,000 per batch per month. Capacity limitations allow a maximum of 3 batches to be produced during each month. In addition, the size of the company's warehouse restricts the ending inventory for each month to at most 3 batches. The company has no initial inventory.

The company wants to find a production plan that will meet all demands on time and minimizes its total production and holding costs over the next 3 months.

- We formulated this problem as a dynamic program with the following shortest path representation:
  - Stage t represents deciding to produce in month t ($t = 1, 2, 3$), or the end of the decision-making process ($t = 4$).
  - Node t_n represents having n batches in inventory at the end of stage t ($n = 0, 1, 2, 3$).



Month	Prod. amt.	Edges	Edge lengths
1	0	$(1_n, 2_{n-1})$ for $n = 1, 2, 3$	$1(n-1)$
1	1	$(1_n, 2_n)$ for $n = 0, 1, 2, 3$	$5 + 2(1) + 1(n)$
1	2	$(1_n, 2_{n+1})$ for $n = 0, 1, 2$	$5 + 2(2) + 1(n+1)$
1	3	$(1_n, 2_{n+2})$ for $n = 0, 1$	$5 + 2(3) + 1(n+2)$
2	0	$(2_n, 3_{n-2})$ for $n = 2, 3$	$1(n-2)$
2	1	$(2_n, 3_{n-1})$ for $n = 1, 2, 3$	$5 + 2(1) + 1(n-1)$
2	2	$(2_n, 3_n)$ for $n = 0, 1, 2, 3$	$5 + 2(2) + 1(n)$
2	3	$(2_n, 3_{n+1})$ for $n = 0, 1, 2$	$5 + 2(3) + 1(n+1)$
3	0	not possible!	
3	1	$(3_n, 4_{n-3})$ for $n = 3$	$5 + 2(1) + 1(n-3)$
3	2	$(3_n, 4_{n-2})$ for $n = 2, 3$	$5 + 2(2) + 1(n-2)$
3	3	$(3_n, 4_{n-1})$ for $n = 1, 2, 3$	$5 + 2(3) + 1(n-1)$

DP for inventory management example

```
In [13]: # Solve this DP using networkx here
         # Create empty graph
         G = nx.DiGraph()

         # Add the stage-state nodes
         for t in range(1, 5):
             for n in range(0, 3):
                 G.add_node((t, n))

         # Add the end node
         G.add_node("end")
```

```python
        # Add edges corresponding to production in month 1
        # 0 batches: green edges
        for n in range(1, 4):
            G.add_edge((1, n), (2, n - 1), length=1*(n - 1))

        # 1 batch: blue edges
        for n in range(0, 4):
            G.add_edge((1, n), (2, n), length=5 + 2*(1) + 1*(n))

        # 2 batches: orange edges
        for n in range(0, 3):
            G.add_edge((1, n), (2, n + 1), length=5 + 2*(2) + 1*(n + 1))

        # 3 batches: purple edges
        for n in range(0, 2):
            G.add_edge((1, n), (2, n + 2), length=5 + 2*(3) + 1*(n + 2))

        # Add edges corresponding to production in month 2
        # 0 batches: green edges
        for n in range(2, 4):
            G.add_edge((2, n), (3, n - 2), length=1*(n - 2))

        # 1 batch: blue edges
        for n in range(1, 4):
            G.add_edge((2, n), (3, n - 1), length=5 + 2*(1) + 1*(n - 1))

        # 2 batches: orange edges
        for n in range(0, 4):
            G.add_edge((2, n), (3, n), length=5 + 2*(2) + 1*(n))

        # 3 batches: purple edges
        for n in range(0, 3):
            G.add_edge((2, n), (3, n + 1), length=5 + 2*(3) + 1*(n + 1))

        # Add edges corresponding to production in month 3
        # 0 batches: not possible!

        # 1 batch: blue edges
        for n in range(3, 4):
            G.add_edge((3, n), (4, n - 3), length=5 + 2*(1) + 1*(n - 3))

        # 2 batches: orange edges
        for n in range(2, 4):
            G.add_edge((3, n), (4, n - 2), length=5 + 2*(2) + 1*(n - 2))

        # 3 batches: purple edges
        for n in range(1, 4):
            G.add_edge((3, n), (4, n - 1), length=5 + 2*(3) + 1*(n - 1))

        # Add edges from last stage to the end node
        for n in range(0, 4):
            G.add_edge((4, n), "end", length=0)

        # Solve DP by solving its shortest path representation using Bellman-Ford
        length, nodes, negative_cycle = bf.bellman_ford(G, source=(1, 0), target="end",
        weight="length")
        print("Shortest path length: {0}".format(length))
        print("Shortest path: {0}".format(nodes))
```

```
Shortest path length: 30
Shortest path: [(1, 0), (2, 0), (3, 1), (4, 0), 'end']
```

- The minimum total production and holding cost over the next 3 months is 30.
- To achieve this minimum cost, the company should produce 1 batch in month 1, 3 batches in month 2, and 3 batches in month 3.